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A Heuristic Method for Nonconvex Optimization in Mechanics: Conceptual Idea, Theoretical Justification, Engineering Applications*

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Abstract. Structures involving nonmonotone, possibly multivalued reaction-displacement or stress-strain laws cannot be effectively treated by the numerical methods for classical non-linearities. In this paper we make use of the fact that these problems have as a variational formulation a hemivariational inequality, leading to a noncovex optimization problem. A new method is proposed which approximates the nonmonotone problem by a series of monotone ones. The method constitutes an iterative scheme for the approximation of the solutions of the corresponding hemivariational inequality. A simple numerical example demonstrates the conceptual idea of the proposed numerical method. In the sequel the method is applied on an engineering problem concerning the ultimate strength analysis of an eccentric braced steel frame.

Key words: Hemivariational inequalities; Nonconvex optimization

AMS: 90C33, 65K10

1. Introduction

Elements involving nonconvex and/or nonsmooth energy potentials appear in several mechanical problems. The nonconvexity of the energy potential appears as a result of the introduction of a nonmonotone, possibly multivalued stress-strain or reaction-displacement law. We mention for example the nonmonotone variant of the well known friction law of Coulomb of Figure 1a which leads to the nonconvex energy potential of Figure 1b. Similar is the situation with the sawtooth stress-strain law of Figure 1c which appears in reinforced concrete under tension (Scanlon's diagram) and leads to the potential energy function of Figure 1d. The same effects may appear also in other problems of structural mechanics.

A common case is that of a frame structure with connections obeying to a nonlinear moment-rotation law. This type of behaviour appears in almost every kind of civil engineering structure (concrete, steel, composite or timber) and is a result of pure or incomplete 'cooperation' between the various structural elements (which is sometimes intentional) and incorporates various local instability effects such as

* This paper is dedicated to the memory of Professor P.D. Panagiotopoulos.

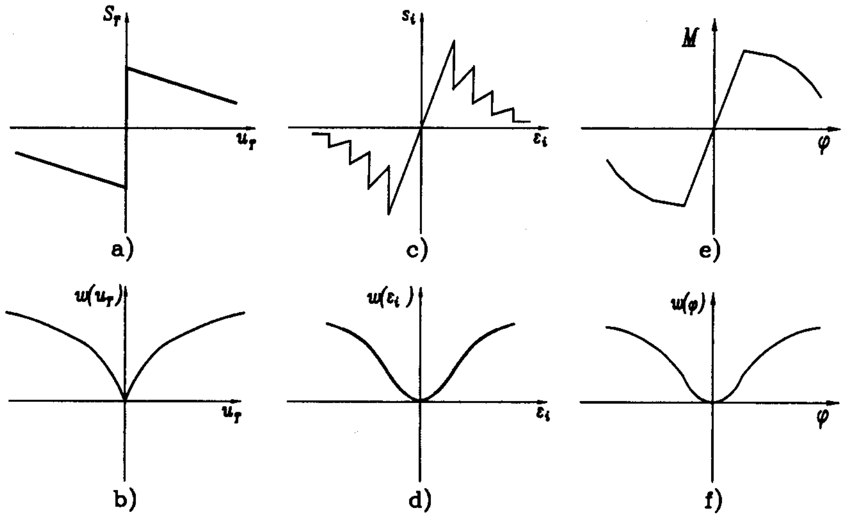

Figure 1. Various nonmonotone laws and the corresponding nonconvex superpotentials.

buckling, crashing and cracking [27]. Figure 1e gives the typical moment-rotation diagram of a steel beam-to-column connection. In most cases the diagram has a significant softening branch leading to the nonconvex energy potential of Figure 1f.

The same type of diagrams relates also the moment with the rotatonal capacity of steel beams, where the softening branch has its nature in the combination of material nonlinearity with severe local buckling of the various parts of the cross-section [7].

The usual problem in structural analysis consists in finding the minimum of the potential function expressing the total work of the system, i.e. that absorbed by the structure minus that supplied by the loading field. When the mechanical system remains in the normal operational framework, i.e. the loads are not severe enough to cause structural damage and no other complex physical mechanisms (material behaviour, unilateral contact, geometric nonlinearity, etc.) contribute to the energy balance of the system, the potential functional is quadratic and the respective unconstrained quadratic minimization problem constitutes a simple, everyday practice problem for the engineering community. In all other cases nonquadratic potentials arise.

Due to this non-quadratic form of the potential function, one is compelled to use a Newton-Raphson type non-linear solution technique. However, there is often the case that the classical methods of linear or linearized analysis encounter forbidding difficulties both in the formulation and in the numerical approximation of problems involving nonmonotone, possibly multivalued stress-strain or reaction-displacement laws such as the ones introduced previously [16, 19, 20, 22]. Then, the respective energy functionals (superotentials) involved, are nonconvex and possibly nonsmooth.

The variational forms for such problems are termed hemivariational inequalities

[19, 20–22]. The respective nonconvex energy functions are called nonconvex superpotentials in accordance to the case of monotone mechanical behaviour described by convex superpotentials, where variational inequalities are obtained [20].

The theory of variational inequalities is closely related to the notion of convex, nondifferentiable superpotentials introduced into mechanics by J.J. Moreau [15] for monotone possibly multivalued boundary conditions and constitutive laws. On the other hand, the hemivariational inequalities were introduced as a generalization of the variational inequalities by P.D. Panagiotopoulos [19, 20, 23]. They are directly related to nonconvex superpotentials and describe nonmonotone boundary or constitutive laws.

The theory of hemivariational inequalities leads to the result that local minima of the potential or the complementary energy of the structure represent equilibrium positions of the problem. It is possible, however, that certain solutions of the problem may not be local minima but other more general types of points which make the potential or the complementary energy ‘substationary’. They are solutions of the differential inclusion $0 \in \bar{\partial}\Pi(u)$, where Π is the potential energy, u is the displacement vector and $\bar{\partial}$ denotes the generalized gradient of Clarke–Rockafellar [2, 25] which can be understood as an extension of the notion of the classical derivative in order to cover nonconvex and nonsmooth functions. Analogous substationarity problems can be formulated in terms of the complementary energy of the structure Π^c and the stresses s, i.e. $0 \in \Pi^c(s)$.

Therefore, the substationarity points obtained constitute all the possible solutions of a hemivariational inequality. This is a generalization of the minimum potential or complementary energy theorems which hold in the case of variational equalities and inequalities [11–14, 20].

The determination of the full set of solutions of a substationarity problem, even when only smooth functionals are involved, remains as yet an open problem and constitutes an area of active research in the field of computational mechanics. This holds indeed for a global optimization problem as well, the latter being only a particular case of the general substationarity problem [4,18].

Moreover, one must keep in mind that the questions posed in mechanics are sometimes different from the ones of global optimization. A global minimum of the potential energy of a structure may be of relatively less importance if a given loading history is not sufficient to drive the structure near this equilibrium point. Stable or unstable equilibria, i.e., local minima or substationary points attainable by the given loading path may be of importance in this latter case.

Although the progress in the theoretical study of the existence and approximation questions for hemivariational inequalities is considerable [9, 17, 23, 24], relatively few methods exist for the numerical treatment [3, 5, 8, 10]. Some of these methods can find only very limited use in practical applications, due to the fact that their effectiveness fails rapidly with increasing problem size (order of one hundred of unknowns), due mainly to numerical stability problems. Moreover, they have inherent difficulties regarding the treatment of the complete vertical branches of the stress-strain or reaction-displacement diagrams.

For all the above reasons, in engineering applications, the sole mathematical examination of the problem has to be abandoned and more heuristic methods have to be investigated [14], which take into account the engineering characteristics of the problems at hand.

The aim of the present paper is to demonstrate a new method for the solution of the nonconvex optimization problems in mechanics and engineering. The characteristic of the developed method is that it reduces the nonconvex optimization problem into a number of convex minimization subproblems. This is equivalent to the approximation of the hemivariational inequality describing the problem involving the nonmonotone laws by a number of variational inequalities that involve monotone laws only. Subsequently, each variational inequality can be treated effectively by specialized convex minimization algorithms [4, 26]. The verified reliability and high convergence rates of the latter make a precious advantage.

2. Description of the heuristic method

In this section a heuristic algorithm is presented for the solution of the substationarity problem of a nonconvex function $\Pi(x)$ which is composed of a convex part $f_c(x)$ and a nonconvex part $w(x)$, i.e.

$$\Pi(x) = f_c(x) + w(x) . \tag{1}$$

This case is very common in mechanics where we deal with problems involving a number of elements with convex energy density and a number of elements with nonconvex energy density.

The substationarity problem of $\Pi(x)$ can be written equivalently in the form of the differential inclusion

$$0 \in \bar{\partial}\Pi(x) \tag{2}$$

where $\bar{\partial}$ denotes the generalized gradient of Clarke–Rockafellar [2, 25], which is an extension of the subdifferential ∂ of convex analysis for nonconvex problems.

The aim of the proposed algorithm is to avoid the nonconvex minimization problem by minimizing a sequence of appropriately defined convex functions in which the nonconvex part $w(x)$ has been replaced by the convex function $p^{(i)}(x)$. Consider first the following minimization problems

$$\min\{\Pi_c^{(i)} = f_c(x) + p^{(i)}(x)\} \tag{3}$$

where in each step the convex function $p^{(i)}(x)$ is selected such that the following relation is fulfilled

$$\partial p^{(i)}(x) = \bar{\partial} w(x) \tag{4}$$

at the point $x^{(i-1)}$. In this case the function $\Pi_c^{(i)}$ is a sum of convex functions and thus a convex function itself.

Then, the nonconvex minimization problem is written in the form

$$\begin{aligned}\min\{\Pi(x)\} &= \min\{f_c(x) + p^{(i)}(x) + [w(x) - p^{(i)}(x)]\} \\ &= \min\{\Pi_c^{(i)} + [w(x) - p^{(i)}(x)]\}\end{aligned} \tag{5}$$

or equivalently, the differential inclusion (2) in the form

$$\begin{aligned}0 \in \bar{\partial}\Pi(x) = \bar{\partial}(f_c(x) + w(x)) &= \bar{\partial}(f_c(x) + p^{(i)}(x) + w(x) - p^{(i)}(x)) \\ &\subset \bar{\partial}(f_c(x) + p^{(i)}(x)) + \bar{\partial}(w(x) - p^{(i)}(x)) \\ &= \partial(f_c(x) + p^{(i)}(x)) + \bar{\partial}(w(x) - p^{(i)}(x)) \\ &= \partial\Pi_c^{(i)} + \bar{\partial}(w(x) - p^{(i)}(x)) .\end{aligned} \tag{6}$$

Based on the previous decomposition of the convex and nonconvex parts, the following approximation scheme can be formulated:

$$\begin{aligned}\min\{\Pi^{(1)}(x)\} &= \min\{\overbrace{\Pi_c^{(1)}(x)}^{1\text{-}step} + \overbrace{[w(x^{(0)}) - p^{(0)}(x^{(0)})]}^{0\text{-}step}\} \\ &= \min\{\Pi_c^{(1)}(x)\} + C^{(1)} \\ &\vdots \\ \min\{\Pi^{(i)}(x)\} &= \min\{\overbrace{\Pi_c^{(i)}(x)}^{i\text{-}step} + \overbrace{[w(x^{(i-1)}) - p^{(i-1)}(x^{(i-1)})]}^{(i-1)\text{-}step}\} \\ &= \min\{\Pi_c^{(i)}(x)\} + C^{(i)} \\ &\vdots \\ \min\{\Pi^{(n)}(x)\} &= \min\{\overbrace{\Pi_c^{(n)}(x)}^{n\text{-}step} + \overbrace{[w(x^{(n-1)}) - p^{(n-1)}(x^{(n-1)})]}^{(n-1)\text{-}step}\} \\ &= \min\{\Pi_c^{(n)}(x)\} + C^{(n)} .\end{aligned} \tag{7}$$

In the above iterative procedure, the first part is referred to the current step, whereas the second part is a scalar number obtained from the solution of the previous step. Using the above decomposition, in each step i, only the convex part $\Pi_c^{(i)}(x) = f_c(x) + p^{(i)}(x)$ is minimized and the obtained solution $x^{(i)}$ is used for the selection of a new convex approximation $p^{(i+1)}(x)$ of the nonconvex function $w(x)$ through (4). This convex function will be used in the next step of the iterative procedure. Also, the constant term $C^{(i+1)}$ is calculated, at the equilibrium point $x^{(i)}$.

If we suppose that in the last case we have the convergence of the iterative scheme, then we have that $|x^{(n)} - x^{(n-1)}| \leqslant \varepsilon_1$, where ε_1 is an appropriately small number. On the assumption that $x^{(n)}$ is not a nondifferentiability point of $w(x)$, then $|C^{(n)} - C^{(n-1)}| \leqslant \varepsilon_2$ where ε_2 is also an appropriately small number. In the case that $x^{(n)}$ is a nondifferentiability point of $w(x)$ then the condition $|x^{(n)} - x^{(n-1)}| \leqslant \varepsilon_1$ is not sufficient to have convergence but the condition $|C^{(n)} - C^{(n-1)}| \leqslant \varepsilon_2$ should also be satisfied.

Finally, we can write for n large enough that

$$\text{argmin}\{\Pi(x)\} = \text{argmin}\{\Pi_c^{(n)}(x)\} \tag{8}$$

where in the left-hand side we mean the local minimum sought.

By means of the previous relations it is easily verified that a solution of the initial minimization problem of $\Pi(x)$ can be obtained using the proposed iterative scheme but the full proof of convergence remains still an open problem. However, in the various numerical experiments we have performed for special forms of convonvex superpotentials, convergence was always achieved in a small number of steps [11, 12, 14].

Here we shall show the graphical meaning of the iterative scheme defined by (3), (4). Let us assume the one-dimensional nonconvex function $\Pi(x) = f_c(x) + w(x)$ which results as a sum of the convex function $f(x)$ and of the nonconvex function $w(x)$ (see Figure 2a). The nonconvex function $w(x)$ results from the integration of the nonmonotone law $g(x)$ (Figure 2b). In an engineering problem this law can be understood, for example, as a nonlinear stress-strain relation or as a nonlinear

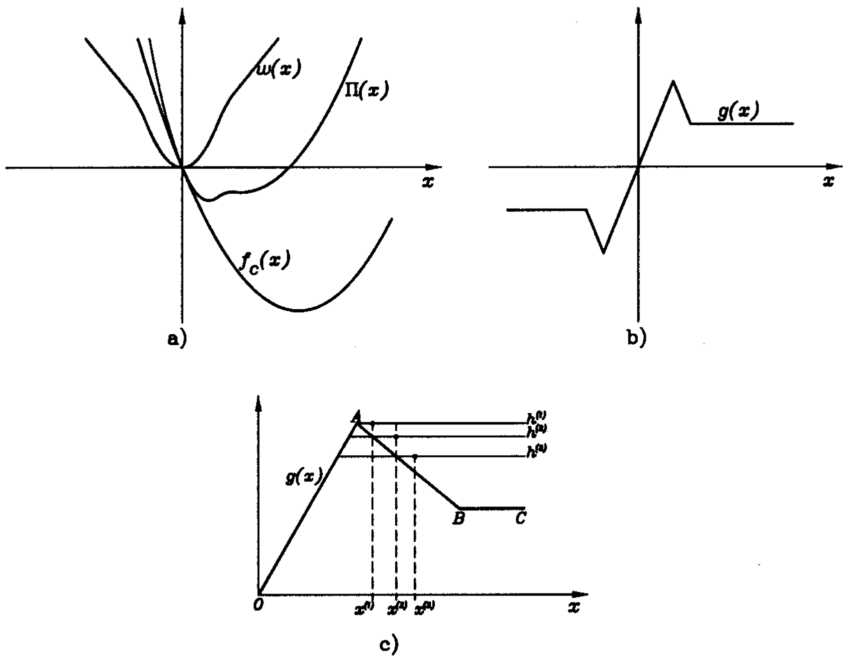

Figure 2. (a) The nonconconvex function $\Pi(x) = f_c(x) + w(x)$. (b) The nonmonotone law corresponding to $w(x)$. (c) Graphical explanation of the heuristic nonconvex minimization algorithm.

boundary condition. In this case, the nonconvex function $w(x)$ is the superpotential of the law $g(x)$.

In the first step we approximate the nonmonotone law g with the fictitious monotone law $h^{(1)}$ of Figure 2c. This monotone law gives rise to the convex function $p^{(1)}$. The minimization of $\Pi_c^{(1)}(x) = f_c(x) + p^{(1)}(x)$ (which is a sum of convex functions) gives as a unique solution the value $x^{(1)}$. This is not a solution of the minimization of $\Pi(x)$ because the solution point does not lie on the nonmonotone law $g(x)$. We select now a new convex function $p^{(2)}$ such that relation (4) is fulfilled. For the one-dimensional case treated here this relation is equivalent to $h^{(2)}(x^{(1)}) = g(x^{(1)})$. One possible such monotone law is the one depicted in Figure 2c. The minimization of $\Pi_c^{(2)}(x) = f_c(x) + p^{(2)}(x)$ yields the solution $x^{(2)}$ and this procedure is continued until the difference between the solutions of two consecutive iterations is small enough. In this case, as it is obvious from (7), the minimizer of $\Pi_c^{(n)}(x)$ will also minimize $\Pi(x)$.

We have to notice here that the possibility presented in Figure 2c for the approximation of the nonmonotone law is only one of the different monotone laws that would approximate the nonmonotone one. In general, the convex superpotentials that approximate the nonconvex superpotential are selected in such a way that the computational effort for the solution of the arising convex problem is minimized. This task depends on the particular nonconvex functions to be approximated.

Thus, the following heuristic algorithm can be formulated:

1. Select a starting point $x^{(0)}$ and initialize i to 1.
2. For the point $x^{(i)}$ select a convex superpotential $p^{(i)}$ such that relation (4) is fulfilled at this point.
3. Find the minimum $x^{(i)}$ of the convex function

$$f_c(x) + p^{(i)}(x) .$$

4. Calculate

$$C^{(i+1)} = w(x^{(i)}) - p^{(i)}(x^{(i)}) .$$

5. If $\|x^{(i)} - x^{(i-1)}\| \leqslant \varepsilon_1$ and $\|C^{(i+1)} - C^{(i)}\| \leqslant \varepsilon_2$ where ε_1 and ε_2 are appropriately small numbers, then a substationarity point of (1) has been determined and terminate the algorithm, else set $i = i + 1$ and repeat step 2.

Given a starting point $x^{(0)}$, the above algorithm is able to detect a substationarity point of $\Pi(x)$ and not only a local minimum point, as it becomes obvious from (6). Different starting points of the iterative procedure, may lead to different substationarity points of $\Pi(x)$.

Concerning the minimization problem of Step 3 of the above algorithm, we have to notice that this convex minimization problem is considered nowadays a trivial task and a lot of efficient algorithms can be applied [4] for its numerical treatment.

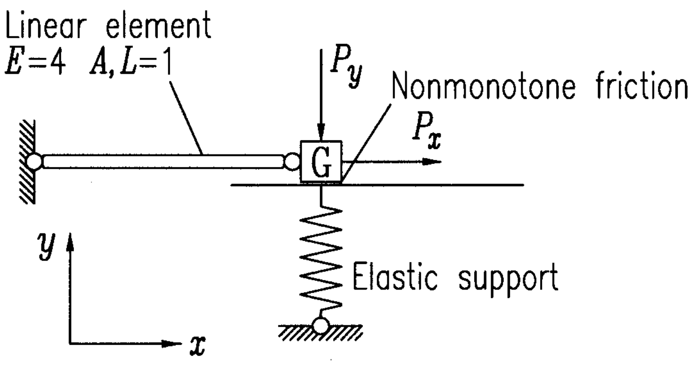

Figure 3. A model unilateral contact problem with nonmonotone friction.

3. A characteristic example

As a first example we will consider the simple structure of Figure 3. The structure consists of the rigid body G which comes in contact with an elastic support. Between the body and the support frictional conditions arise for which we assume that the nonmonotone law of Figure 3 holds. Moreover, the body is connected with an elastic element with a rigid support. The modulus of elasticity for this element is $E = 4$ and has unit length L and cross-section area A. The spring constant for the elastic support is $k = 8$. The structure is loaded with the forces $P_x = 1$ and $P_y = -1$ at G. For the given loading we want to determine the possible equilibrium positions of the system.

With the given data we formulate the potential energy of the structure as a function of the displacements u_x, u_y of the point G. The total potential energy Π_{nc} is written as the sum of the potential energy of the linear element and of the elastic support and of the superpotential of the frictional joint, i.e.

$$\Pi_{nc} = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} + j(u_x) - \mathbf{u}^T\mathbf{P}\,, \quad u_y \leqslant 0 \tag{9}$$

where $\mathbf{u} = [u_x \;\; u_y]^T$, $\mathbf{P} = [P_x \;\; P_y]^T$ and $\mathbf{K}$ is the stiffness matrix of the structure which has the very simple form $\mathbf{K} = [\begin{smallmatrix} 4 & 0 \\ 0 & 8 \end{smallmatrix}]$.

Moreover, the nonconvex superpotential $j(.)$ has the form depicted in Figure 4b. The nonconvex function Π_{nc}, for $u_x \in [-0.1, 0.5]$ and $u_y \in [0, -0.25]$ is depicted in Figure 5a. As it is more clear in Figure 5b, the function has 3 substationarity points, i.e. 1 global minimum, 1 local minimum and 1 saddle point. We notice that even with two unknowns, the situation can be very complicated. Now we will see how the proposed algorithm can approximate the above substationarity points. In this problem, in order to approximate the nonmonotone law we select monotone laws of the type depicted with the dashed line in Figure 3a (Coulomb friction laws).

Starting the algorithm from the point $(u_x, u_y) = (0, 0)$ we obtain as a solution the global minimum $(0, -0.125)$. Defining other starting points, the algorithm con-

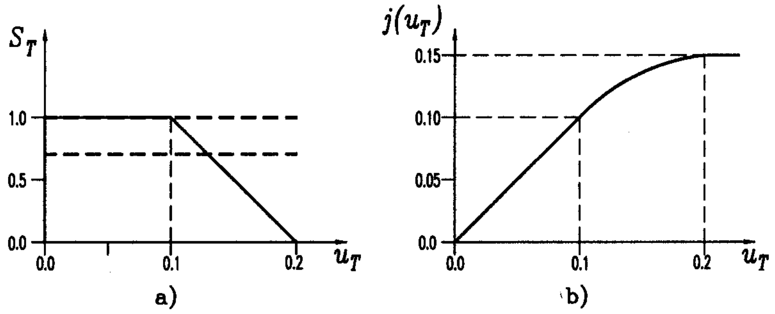

Figure 4. The adopted friction law and the respective nonconvex superpotential.

verges to other solutions. Thus, if we start from the point (0.2, 0.00) we obtain as a solution the point (0.25, −0.125).

Although the algorithm was able to determine the global and the local minimum, it was impossible to obtain as solution the saddle point. This is due to the fact that the proposed method approximates the nonconvex functional with convex ones, thus it has the inherent property to approximate only substationarity points that give adequate convexity information in their neighborhood. Under these circumstances, it is impossible to obtain as a solution local maxima and saddle points, unless the starting point coincides with one of them. From the engineering point of view this result seems acceptable as these substationarity points are not stable solutions of the problem.

4. Application to structures involving softening elements

In this section we will study the case of a structure with elements obeying to nonlinear moment-rotation laws.

Without loosing generality let us assume a structure Ω consisting of bar elements. For the material of m of these elements we assume that the classic elastoplastic law of Figure 6a holds. This law yields the convex superpotential $q(\varphi)$ of Figure 6b.

Moreover, let us assume that the structure contains l elements exhibiting a softening behaviour, i.e. a nonmonotone law $M = g(\varphi)$ (Figure 6c) relates the moment M with the rotational capacity φ. In this case the respective nonconvex superpotential $\tilde{w}(\varphi)$ is one-dimensional and $\tilde{w}(\xi)$ is the area between the horizontal axis and the graph until the point ξ of the horizontal axis, i.e. $\tilde{w}(\xi) = \int_0^{\xi} g(\varphi)\, d\varphi$ (see Figure 6d).

We define now the set $\mathscr{S}_1$ consisting of the m elements of the first category and the set $\mathscr{S}_2$ consisting of the l elements of the second category. For this problem, the principle of virtual work is written in the form:

$$\mathbf{s}^T(\mathbf{e}(\mathbf{u}^*) - \mathbf{e}(\mathbf{u})) = \mathbf{p}^T(\mathbf{u}^* - \mathbf{u}) \quad \forall \mathbf{u}^* \in V_{ad} \tag{10}$$

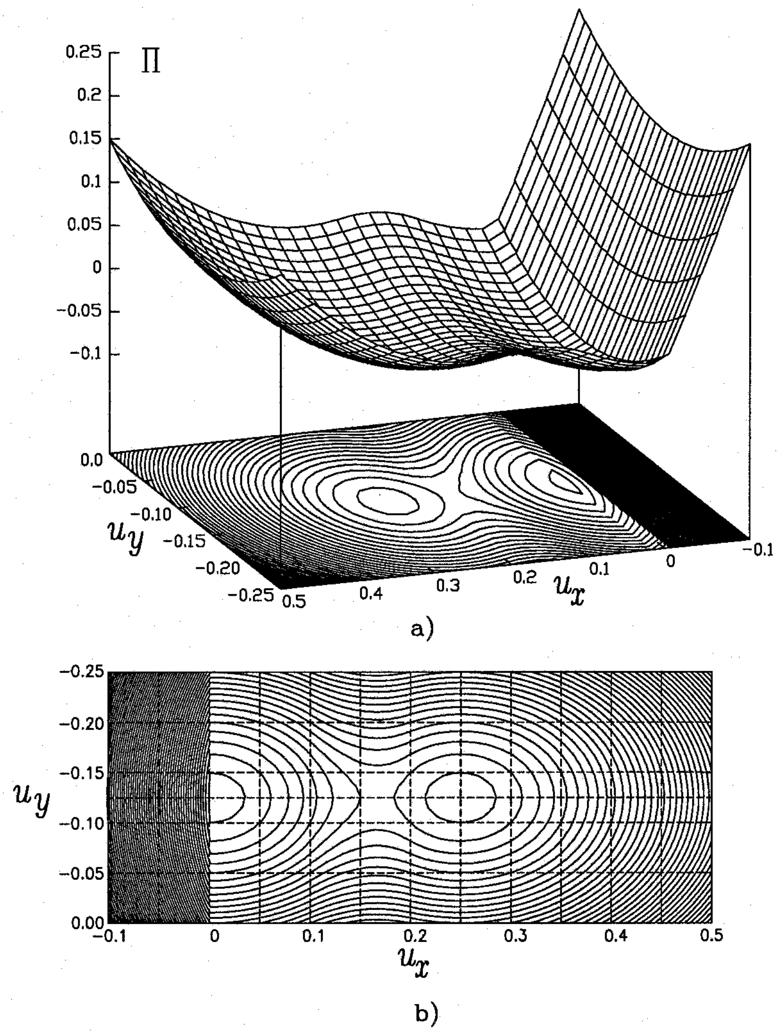

Figure 5. (a) The nonconvex potential energy function of the structure. (b) Isolines of the potential energy function.

where **s**, **e**, **p**, **u** are the stress, strain, loading and displacement vectors, respectively, and V_{ad} is the kinematically admissible set (i.e. fixed nodes have $u_i = 0$). Suppose now that **s** (resp. **e**) consists of three parts $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$ (resp. $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$) where $\mathbf{s}_1$ (resp. $\mathbf{e}_1$) corresponds to the moments (resp. the rotations) of the elements of set $\mathscr{S}_1$, $\mathbf{s}_2$ (resp. $\mathbf{e}_2$) corresponds to the moments (resp. the rotations) of the elements of set $\mathscr{S}_2$ and $\mathbf{s}_3$ (resp. $\mathbf{e}_3$) corresponds to the rest stresses (resp. strains) of both sets.

Equation (10) is now written in the form

$$\mathbf{s}_1^T(\mathbf{e}_1(\mathbf{u}^*) - \mathbf{e}_1(\mathbf{u})) + \mathbf{s}_2^T(\mathbf{e}_2(\mathbf{u}^*) - \mathbf{e}_2(\mathbf{u})) + \mathbf{s}_3^T(\mathbf{e}_3(\mathbf{u}^*) - \mathbf{e}_3(\mathbf{u}))$$
$$= \mathbf{p}^T(\mathbf{u}^* - \mathbf{u}) \quad \forall \mathbf{u}^* \in V_{ad} \,. \tag{11}$$

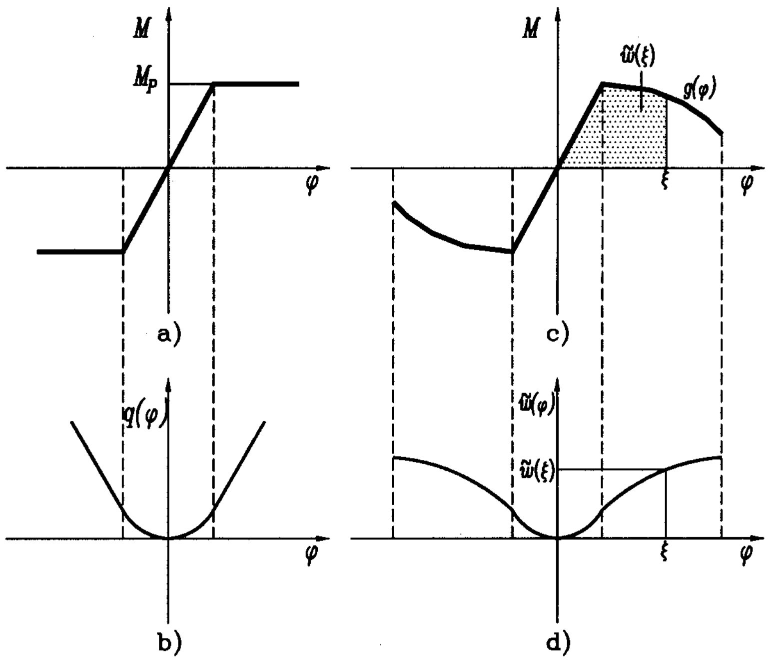

Figure 6. Monotone and nonmonotone moment-rotation relationships and the corresponding superpotentials.

For the m elements of $\mathscr{S}_1$ we can write $s_{1k} \in \partial q(e_{1k})$, $k = 1, \ldots, m$ which is equivalent to the variational inequality

$$qe_{1k}(u^*)) - q(e_{1k}(u)) \geqslant s_{1k}^T(e_{1k}(u^*) - e_{1k}(u)) \,, \quad k = 1, \ldots, m \quad \forall u^* \in \mathbb{R} \,. \tag{12}$$

Similarly, for the l elements of $\mathscr{S}_2$ we can write $s_{2j} \in \bar{\partial}\tilde{w}(e_{2j})$, $j = 1, \ldots, l$ which is equivalent to the hemivariational inequality

$$\tilde{w}^0(e_{2j}(u^*) - e_{2j}(u)) \geqslant s_{2j}^T(e_{2j}(u^*) - e_{2j}(u)) \,, \quad j = 1, \ldots, l \quad \forall u^* \in \mathbb{R} \tag{13}$$

where $\tilde{w}^0$ is the directional derivative [23] of the nonconvex superpotential $\tilde{w}$.

Using the inequalities (12), (13), Equation (11) is transformed in the form

$$\sum_{k=1}^{m} [q(e_{1k}(\mathbf{u}^*)) - q(e_{1k}(\mathbf{u}))] + \sum_{j=1}^{1} \tilde{w}^0(e_{2j}(\mathbf{u}^*) - e_{2j}(\mathbf{u})) + \mathbf{s}_3^T(\mathbf{e}_3(\mathbf{u}^*) - \mathbf{e}_3(\mathbf{u})) \geqslant \mathbf{p}^T(\mathbf{u}^* - \mathbf{u}) \quad \forall \mathbf{u}^* \in V_{ad} \,, \tag{14}$$

which is a variational-hemivariational inequality. The corresonding substationarity problem reads:

Find $\mathbf{u} \in V_{ad}$ such that the potential energy

$$\Pi_{nc}(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} + \sum_{k=1}^{m} q(e_{1k}(\mathbf{u})) + \sum_{j=1}^{l} \tilde{w}(e_{2j}(\mathbf{u})) - \mathbf{p}^T\mathbf{u} \tag{15}$$

is substationary, where $\mathbf{K} = \mathbf{G}_3\mathbf{K}_0\mathbf{G}_3^T$, $\mathbf{K}_0$ is the inverse of the natural flexibility matrix $\mathbf{F}_0$ and $\mathbf{G}_3$ is the equilibrium matrix that corresponds to $\mathbf{s}_3$.

The solution of the above problem can be found by applying the heuristic nonconvex optimization approach presented in Section 2. Using approximating monotone laws having the same shape with the classical plasticity law of Figure 6a, the problem arising in each step of the iterative procedure is a classical plasticity problem which can be treated numerically by the classical plasticity methods of structural analysis.

5. Numerical application

As an example we will consider the multistorey eccentric braced plane frame of Figure 7. The structure consists of HEB300 columns and IPE220 beams. The diagonals are HEA100 elements. In the case of seismic loading, this kind of frames shall be designed so that the beams connected to the diagonals are able to dissipate energy by the formation of plastic bending mechanisms. Therefore, the adoption of different moment-rotation curves for these beams may result to different ultimate strength for the whole frame.

For the rotational capacity of the beams of the structure the general moment-rotation curve of Figure 8a is assumed to hold while for the columns the classical elastoplastic diagram is used. In the diagram of Figure 8a, the parameters m, φ_y, φ_u, K_h, K_d are calculated according to the works of Kato and Akiyama [1, 6].

In the particular case treated here, due to the lack of torsional restraint of the

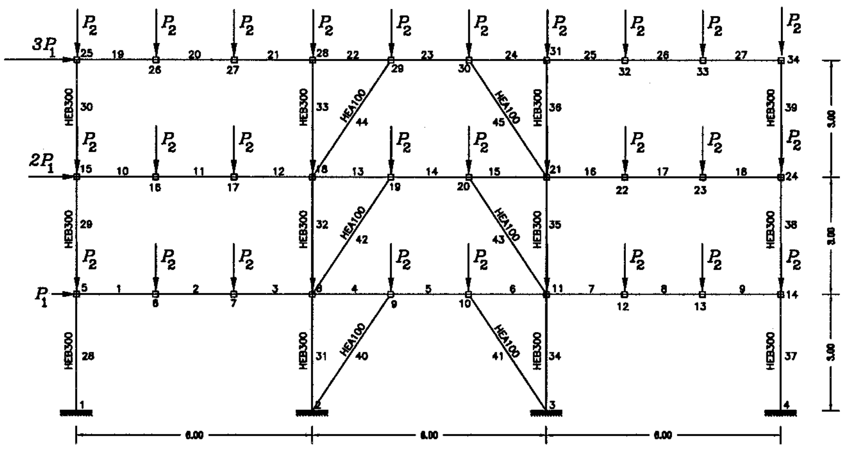

Figure 7. Multistorey eccentric braced frame.

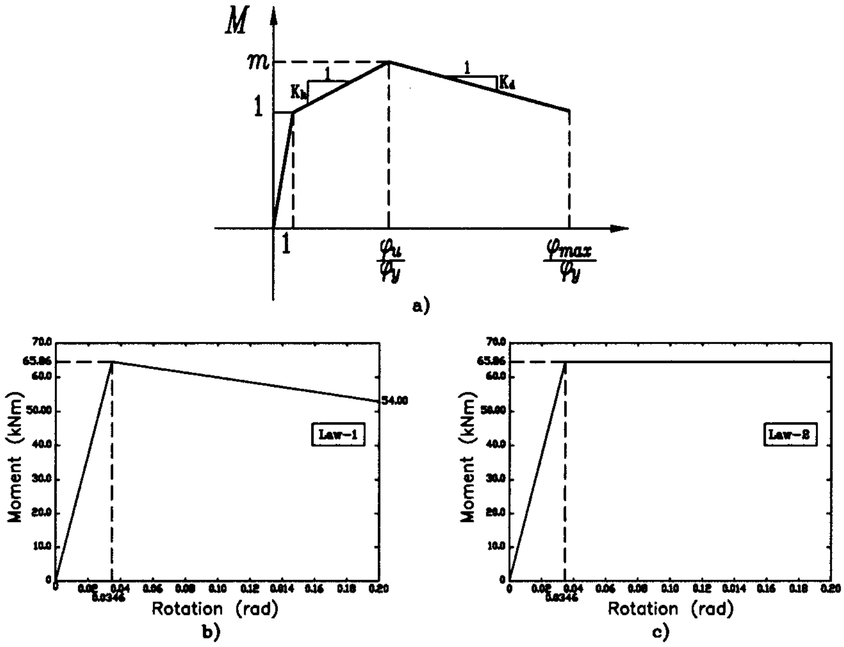

Figure 8. (a) General moment-rotation relationship. (b, c) The adopted moment-rotation laws.

beams, flexural-torsional buckling has to be taken into account. According to the previously mentioned literature, this effect, together with the local buckling effect has as a result the reduction of the parameter m. In the case that the calculated parameter m is less than 1, the strain-hardening branch of the diagram of Figure 8b is eliminated. Also, the coupling of the buckling modes leads to a higher slope of the softening branch [1, 6]. The specific geometrical and elastic data used here lead to the moment-rotation diagram of Figure 8b (law f_1). For comparison reasons, the simple law of Figure 8c is also considered (law f_2) which is similar to the classical plasticity law. The steel grade is Fe360 with a yield stress of $235\,\mathrm{N/mm^2}$.

The loading of the structure is shown in Figure 7, where $P_1 = 30.0\,\mathrm{kN}$ and $P_2 = 30.0\,\mathrm{kN}$. Forty load cases are considered, by multiplying the horizontal loads with the factors of Table 1, while the vertical loads remain constant.

The structure is analyzed using the algorithm of the previous section. Figure 9a

Table 1. The considered load cases

Load case	Factor	Load case	Factor
5	1.25	25	2.25
10	1.50	30	2.50
15	1.75	35	2.75
20	2.00	40	3.00

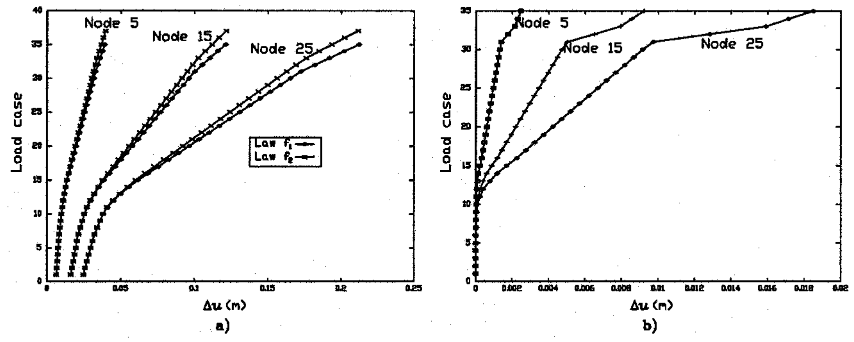

Figure 9. (a) Load-displacement curves for characteristic nodes of the frame. (b) Differences of the displacements resulting from the assumption of laws f_1 and f_2, with respect to the load.

gives the horizontal displacements for selected nodes of the structure, for the assumption of the two different moment-rotation curves. As the horizontal loading increases, more and more beams enter into the softening region of the moment-rotation curve. This phenomenon has as a result the gradual reduction of the beams' moments, the redistribution of the stresses of the whole frame, and the increase of the moments in the columns. Further increase of the horizontal loading has as a result the plastification of the lower sections of the columns. Finally, as the plastification procedure develops under increasing loading, the whole frame becomes kinematically unstable and collapses at load case 35. Analogous is the situation if we assume that the moment-rotation law of Figure 8c holds. In this case, the elastoplastic diagram has as a result the constant value of the moment, after reaching the plastification moment M_p. Thus, the redistribution of the stresses is slower due to the increased rotational capacity of this law and the frame collapses at load case 38. Therefore, the collapse load for law f_2 is about 6% higher than the collapse load for law f_1.

As it was expected, the assumption of the softening moment-rotation curve results to larger horizontal displacements. Figure 9b gives the differences Δu of the displacements that correspond to the assumption of law f_1 from the ones that correspond to law f_2, with respect to the various load cases. The results are depicted for the characteristic nodes 5, 15, 25. It is noticed that the differences of the displacements increase together with the loading.

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